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On the Trend to Equilibrium for Some Dissipative Systems with Slowly Increasing *a Priori* Bounds

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We prove convergence to equilibrium with explicit rates for various kinetic equations with relatively bad control of the distribution tails: in particular, Boltzmann-type equations with (smoothed) soft potentials. We compensate the lack of uniform-in-time estimates by the use of precise logarithmic Sobolev-type inequalities, and the assumption that the initial datum decays rapidly at large velocities. Our method not only gives explicit results on the times of convergence, but is also able to cover situations in which compactness arguments apparently do not apply (even mere convergence to equilibrium was an open problem for soft potentials).

KEY WORDS: Fokker–Planck equation; Landau and Boltzmann equations with soft potentials; logarithmic Sobolev inequalities; decay of relative entropy.

1. INTRODUCTION

We consider in this work the problem of trend to equilibrium for collisional kinetic equations of the form

$$\frac{\partial f}{\partial t} = Q(f) \tag{1}$$

where the unknown $f(t, v) \geqslant 0$ ($t \geqslant 0$, $v \in \mathbb{R}^N$) is a probability density on $\mathbb{R}^N_v$, and Q is a collision operator which is mass-preserving and *dissipative*, in the sense that solutions of (1) make a certain entropy functional decrease with time. We shall mainly be interested in situations where the interaction modelled by Q is rather "weak"—soft interaction potentials in

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the usual terminology. Before explaining in detail what we mean by this, let us give more background on the problem.

We shall consider three collision operators:

— the linear Fokker–Planck operator (see ref. 17)

$$Lf = \nabla_v \cdot (\nabla_v f + f \,\nabla W) \tag{2}$$

where W is a potential on $\mathbb{R}^N$ satisfying

$$\int_{\mathbb{R}^N} e^{-W(v)}\, dv = 1$$

— the nonlinear (Fokker–Planck-) Landau operator (see ref. 20 and the references therein)

$$Q_L(f, f) = \nabla_v \left\{ \int_{\mathbb{R}^N} dv_*\, a(v - v_*)[f_*\, \nabla f - f(\nabla f)_*] \right\} \tag{3}$$

with $\phi_* \equiv \phi(v_*)$ and

$$a(z) = \Psi(|z|)\, \Pi(|z|), \qquad \Pi_{ij}(z) = \delta_{ij} - \frac{z_i z_j}{|z|^2}$$

— the Boltzmann operator[8, 9]

$$Q(f, f) = \int_{\mathbb{R}^N} dv_* \int_{S^{N-1}} d\sigma\, B(v - v_*, \sigma)(f' f'_* - f f_*) \tag{4}$$

with $f' = f(v')$ and so on,

$$\begin{cases} v' = \dfrac{v + v_*}{2} + \dfrac{|v - v_*|}{2}\,\sigma \\[2ex] v'_* = \dfrac{v + v_*}{2} - \dfrac{|v - v_*|}{2}\,\sigma \end{cases} \tag{5}$$

and $B(z, \sigma)$ is a nonnegative function depending only on $|z|$ and $(z/|z|, \sigma)$.

For the first operator, with the unique exception of $W(v) = |v|^2/2$, there is only one conservation law (the mass $\int f\, dv$). The steady state is the probability distribution e^{-W}, and there is a variety of entropies, given by

$$\int_{\mathbb{R}^N} \phi(f e^W)\, e^{-W}$$

where $\phi(s)$, $s \geqslant 0$ is a (strongly) convex function. In the following we will consider only the Kullback relative entropy, that corresponds to the choice $\phi(s) = s \log s$

$$H(f \mid e^{-W}) = \int_{\mathbb{R}^N} f(\log f + W)$$

For the other two models, there are two additional conservation laws: momentum and kinetic energy, i.e., $\int fv\, dv$, $\int f\,|v|^2/2\, dv$. Thus we may assume without loss of generality that

$$\int_{\mathbb{R}^N} f(v)\, v\, dv = 0, \qquad \int_{\mathbb{R}^N} f(v)\, |v|^2 = N \tag{6}$$

and then the steady state is the centered gaussian (or Maxwellian)

$$M(v) = \frac{e^{-|v|^2/2}}{(2\pi)^{N/2}}$$

while the entropy is again $H(f \mid M)$. By (6), actually

$$H(f \mid M) = H(f) - H(M)$$

where H is Boltzmann's H-functional,

$$H(f) = \int_{R^N} f \log f$$

Contrary to the linear case, for such models this is usually the only entropy functional, see ref. 16.

The precise study of the trend to equilibrium for all three equations has received much attention. While the study of the linear Fokker–Planck equation is relatively old, it is only recently that precise estimates (by this we mean entirely explicit) have been obtained for the operators (3) and (4): see refs. 11 and 19 respectively. These works were strongly influenced by the pioneering contribution of Carlen and Carvalho.[6, 7]

The methods are based on establishing certain differential inequalities of the form

$$D(f) \geqslant C(f)\, H(f \mid f_\infty)^\alpha \tag{7}$$

Here f_∞ stands for the steady state, so that $H(f \mid f_\infty)$ is the entropy functional, $D(f)$ stands for the entropy dissipation in the model on consideration, and $C(f)$ is a positive constant depending on a priori estimates of f. If one can establish an a priori bound on $C(f)$, the inequality (7) entails that the relative entropy $H(f \mid f_\infty)$ satisfies a differential inequality $-\dot{H} \geqslant CH^\alpha$. This implies immediately that it goes to 0 with an explicit rate (exponential if $\alpha = 1$, algebraic if $\alpha > 1$).

Estimates of the form (7) have been established for all three interaction operators (2), (3), (4), under some assumption of "strong interaction." Namely,

— for the Fokker–Planck operator,

$$D(f) \geqslant \frac{1}{2\lambda} H(f \mid e^{-W}) \qquad \text{if} \quad D^2 W \geqslant \lambda \,\mathrm{Id}$$

this is the standard logarithmic Sobolev inequality of Bakry and Emery.[4]

— for the Landau operator,

$$D(f) \geqslant C(f)\, H(f \mid M) \qquad \text{if} \quad \Psi(|z|) \geqslant K\, |z|^2, \quad K > 0$$

here $C(f)$ is a constant depending only on (say) $H(f)$, see ref. 11.

— for the Boltzmann operator,

$$D(f) \geqslant C_\varepsilon(f)\, H(f \mid M)^{1+\varepsilon} \qquad \text{if} \quad B(z, \sigma) \geqslant K$$

where $C_\varepsilon(f)$ depends on some moments of f (of order greater than $4 + 2/\varepsilon$, some moments of $f \log f$ (of order greater than $2 + 2\varepsilon$), and a local lower bound for f, for instance of the form $f \geqslant K e^{-A\,|v|^2}$; see ref. 19.

In all three cases, there are also perturbation lemmas which allow to cover the case when the interaction is strong for "most" of the phase space. For the Fokker–Planck operator, this will mean that $D^2 W \geqslant \lambda \,\mathrm{Id}$ out of a compact set, while for the other two models this will mean that the functions $\Psi(|z|)/|z|^2$ or $B(z, \sigma)$ vanish on a set of zero measure (in the first case, exponential decay still holds, while in the other two, only algebraic decay is proven). In the language of kinetic theory, this means that existing proofs typically cover the case of *hard potentials*.

The question we want to examine here is precisely how to get estimates of trend to equilibrium if the interaction is *weak*. For our three models, this means typically that $D^2 W$, $\Psi(|z|)/|z|^2$ or $B(z, \sigma)$ tend to 0 as $|z| \to \infty$, with an algebraic decay.

Let us first point out that even *convergence* to equilibrium (without any explicit estimates) is not a priori clear in this situation. In order to get a better feeling of the difficulty, let us prove convergence for the Fokker–Planck equation

$$\frac{\partial f}{\partial t} = \nabla_v \cdot (\nabla_v f + f \, \nabla W) \tag{8}$$

with standard PDE arguments. By conservation of mass and decrease of the relative entropy, the family $f(t, \cdot))_{t \geqslant 0}$ is weakly compact in L^1. Hence if (t_n) is any sequence of times going to infinity, we find (taking subsequences if necessary) that

$$f(t_n + \cdot, \cdot) \to g \qquad \text{in } w - L^p([0, T], L^1(\mathbb{R}^N))$$

for all $1 \leqslant p < \infty$, $T > 0$, where $g(t, \cdot)$ is a probability density. By convexity of the entropy dissipation functional (see its explicit form in Section 2), we find $\int_0^T D(g(t, \cdot)) \, dt = 0$, which implies that g is identically equal to the steady state. We conclude that f converges weakly to the steady state as time goes on, and in fact strongly because the family $(f_t)_{t \geqslant 1}$ satisfies uniform smoothness bounds.

The utility of this method is severely limited. First, it relies on compactness arguments: while these arguments actually succeed in proving convergence, they do not rule out the possibility that it may be so slow as to be physically irrelevant in the context of statistical mechanics (like Poincaré's recurrence theorem—as recalled by an anonymous referee). From the point of view of the physics, it is not very satisfying, because it does not reflect the characteristics of the interaction, and tells us nothing about the way the trend to equilibrium is affected by the behaviour of W. Moreover, it is not robust, in the sense that in some situations, it may not extend to the nonlinear case. Indeed, consider for instance the Boltzmann equation $\partial_t f = Q(f, f)$. To conclude by a similar argument we need not only know that the weak limit g is a probability density, but also that it satisfies the same conditions of moments (6) than f. This imposes to know *a priori tightness* of the second moment of f. In the case of hard potentials, such estimates are easy to get as a consequence of uniform boundedness of higher-order moments; but in the case of so-called soft potentials ("weak" interaction as presented above), obtaining uniform bounds on any moment higher than 2 is an open problem. This difficulty is well-identified since the work of Desvillettes.[10] Using compactness arguments, and bounds on moments that grow linearly in time, Desvillettes was able to prove the very weak result that for (not too) soft potentials, there is at least one sequence of times (t_n), $t_n \to \infty$, such that $f(t_n) \rightharpoonup M$.

In this paper, we shall improve by far this result, and actually give an explicit estimate of strong convergence to equilibrium, provided that the initial datum decays rapidly at infinity, and is sufficiently integrable. We shall do so without establishing any new uniform a priori estimate. Rather, we shall compensate the lack of uniform estimates by the use of modified versions of the logarithmic Sobolev-type inequalities stated above. This method will require only a priori bounds (on some weighted norms of f) that *do not increase too fast with time*.

As we shall show, such bounds are often (rather) easy to obtain, and satisfying conclusions will follow, maybe at the price of imposing rather strong conditions on the initial datum. We shall illustrate this approach in the three next sections, respectively on the Fokker–Planck, Landau, and Boltzmann equations.

In short, our main results show that for soft potentials, there is trend to equilibrium, with an algebraic rate which can be very good if the lack of collisions is compensated by a rapid decay of the initial datum. This result is not so natural: one could have expected that the degeneracy of soft potentials for large relative velocities resulted in a rather bad rate. We mention that numerical simulations usually show a somewhat slower decay to equilibrium for soft potentials, than for hard potentials—yet there is no conclusive evidence up to our knowledge.

Let us discuss now the range of application of our method. For the Fokker–Planck equation (8), we are able to cover essentially all of the natural range of parameters, that is

$$W(v) \sim c\,|v|^{\alpha} \qquad \text{as} \quad |v| \to \infty, \qquad 0 < \alpha < 2 \tag{9}$$

For $\alpha \leqslant 0$, the condition $e^{-W} \in L^1$ is not satisfied any more, and the problem becomes meaningless.

For the two nonlinear models given by (3) and (4), we are only able to treat a range of interaction which corresponds formally to (9):

$$\frac{\Psi(|z|)}{|z|^2} \sim c\,|z|^{\gamma}, \qquad B(z,\sigma) \sim c\,|z|^{\gamma}\, b\left(\frac{z}{|z|}\cdot\sigma\right)$$

with

$$0 < \gamma + 2 < 2 \tag{10}$$

The range of exponents given by (10) was identified in ref. 20 for a different problem, and there called *moderately soft potentials*. There, it was shown that (at least from the technical point of view) $\gamma = -2$ is a limit exponent as concerns the singularity of the cross-section B for $z=0$. Here, it will turn out that this is also a limit exponent for the *tail behaviour*.

When $\gamma < -2$, we enter the domain of *very soft potentials*, so far rather mysterious. This is a particular feature of the nonlinear Boltzmann and Landau equations, it does not occur for the Fokker–Planck equation. As concerns the problem of the singularity at the origin, weak (renormalized) solutions are built in ref. 1 for $-N \leqslant \gamma \leqslant -2$, and "stronger" weak solutions in ref. 21 for $-4 < \gamma \leqslant -2$. From the point of view of the physics, the most interesting case is the Landau equation with $N=3$, $\gamma = -3$, which corresponds to Coulomb interaction. Here our problem is not to overcome the singularity at the origin, but the degeneracy at infinity; therefore we shall work with smoothed versions of the very soft potentials, that is replace $|z|^\gamma$ by $(1+|z|)^\gamma$. Even taking this into account, we are unable to cover the case $\gamma = -3$ at present, though we manage to treat any exponent $\gamma > -3$! Yet it seems that a more careful procedure may enable to include this limit case, though with a very bad (logarithmic!) rate, as we shall see.

2. THE FOKKER–PLANCK EQUATION WITH WEAK DRIFT

In this section, we study the trend to equilibrium for the Fokker–Planck equation (8), where W lies in $W^{2,\infty}_{\text{loc}}$, $\int e^{-W}=1$, and W is degenerately convex at infinity, in the sense

$$U(v) - a \leqslant W(v) \leqslant U(v) + b \tag{11}$$

Here a, b are nonnegative constants and U is convex degenerate,

$$D^2 U(v) \geqslant \lambda(|v|) = c(1+|v|)^{\alpha-2}, \qquad c>0, \quad \alpha \in (0,2) \tag{12}$$

Without loss of generality we assume that U takes its unique minimum at 0, so that U satisfies a bound below proportional to $|v|^\alpha$.

Condition (12) holds true for a radial potential $U(v) = u(|v|)$ if

$$\min\left[u''(r), \frac{u'(r)}{r}\right] \geqslant c r^{\alpha-2}$$

So the "typical" case is $W(v) = |v|^\alpha + C$. In dimension 1, for such a potential, a logarithmic Sobolev inequality does not hold if $\alpha < 2$: see the criterion of Bobkov and Götze,[5] and also the study ref. 3. Therefore, we first establish a *modified logarithmic Sobolev inequality*, where we compensate the lack of convexity of W by the use of moments of f.

We use the notations

$$M_s(f)=\int_{\mathbb{R}^N} f(v)(1+|v|^2)^{s/2}\,dv,$$
$$H(f\,|\,g)=\int f\log\frac{f}{g},\qquad I(f\mid g)=\int f\left|\frac{\nabla f}{f}-\frac{\nabla g}{g}\right|^2 \tag{13}$$

and we recall that $I(f\mid g)$ is the relative Fisher information of f with respect to g.

Proposition 1. Let $W\in W^{2,\infty}_{loc}$ satisfy the assumptions (11), (12). Then there exists a constant C, depending only on a, b, c, α, such that for all $s\geqslant 2$,

$$H(f\,|\,e^{-W})\leqslant CI(f\,|\,e^{-W})^{1-\delta}\,M_s(f)^{\delta}$$
$$\delta=\delta(s)=\frac{2-\alpha}{2(2-\alpha)+(s-2)}\in(0,1/2) \tag{14}$$

Remark. As $\alpha\to 2$, we recover the usual logarithmic Sobolev inequality.

Proof. To each real number $R\geqslant 1$ one associates the auxiliary potentials $\widetilde{W}=\widetilde{W}_R$, defined by

$$\widetilde{W}(v)=W(v)+\frac{\lambda(R)}{2}\left(|v|-\frac{R}{2}\right)^2 1_{|v|\geqslant R/2}+C_R$$

where C_R is a normalization constant such that $\int e^{-\widetilde{W}_R}=1$.

Clearly,

$$\widetilde{U}-a\leqslant\widetilde{W}_R\leqslant\widetilde{U}+b$$

with

$$\widetilde{U}(v)=U(v)+\frac{\lambda(R)}{2}\left(|v|-\frac{R}{2}\right)^2 1_{|v|\geqslant R/2}+C_R$$

For $|v|\leqslant R$, of course $D^2\widetilde{U}_R\geqslant\lambda(R)\,Id$, while for $|v|\geqslant R$, one has $|v|-R/2\geqslant|v|/2$, and the Hessian of the added potential is bounded below by $\lambda(R)/2$ Id. Therefore, $\tilde{U}_R$ is (uniformly) strictly convex. This enables to apply to $\tilde{W}_R$ the Holley–Stroock perturbation lemmas[3, 13] to find

$$H(f \mid e^{-\tilde{W}}) \leqslant \frac{e^{a+b}}{\lambda(R)} I(f \mid e^{-\tilde{W}}) \tag{15}$$

Next, we shall convert (15) into an estimate for W. First, let us estimate from below $\tilde{W} - W$: clearly, $\tilde{W} - W \geqslant C_R$, and

$$e^{C_R} = \int e^{-W(v)} e^{-(\lambda(R)/2)(|v| - R/2)^2 1_{|v| \geqslant R/2}} \, dv$$

$$= \int_{|v| \leqslant R/2} e^{-W(v)} \, dv \int_{|v| \geqslant R/2} e^{-W(v)} e^{-(\lambda(R)/2)(|v| - R/2)^2} \, dv$$

For $R = 0$, $0 < e^{C_0} = \int_{\mathbb{R}^N} e^{-W(v)} e^{-(c/2)\,|v|^2} \, dv < 1$, while for $R = R_1$ large enough (depending on a and the lower bound for U), $\int_{|v| \leqslant R_1/2} e^{-W} \, dv = \frac{1}{2}$. This implies

$$0 < \inf_{R \leqslant R_1} e^{C_R} = d(R_1) < 1$$

Hence, for $R \leqslant R_1$

$$C_R \geqslant -\log \frac{1}{d(R_1)} \geqslant -2 \log \frac{1}{d(R_1)} \int_{|v| \leqslant R/2} e^{-W} \, dv \tag{16}$$

where in (16) we used the inequality $2 \int_{|v| \leqslant R/2} e^{-W} \, dv \geqslant 1$, $R \leqslant R_1$.

If $R \geqslant R_1$ we use the lower bound

$$e^{C_R} \geqslant \log \left(1 - \int_{|v| > R/2} e^{-W(v)} \, dv \right) \geqslant -2 \int_{|v| > R/2} e^{-W(v)} \, dv$$

Thus we find (this is a crude estimate)

$$\int f(\log f + W) < \int f(\log f + \tilde{W}) + d \int_{|v| > R/2} e^{-W(v)} \, dv$$

where we denoted $d = \max\{2, 2\log(1/d(R_1))\}$. On the other hand, we write

$$\int f \left| \frac{\nabla f}{f} + \nabla \tilde{W} \right|^2 \leqslant 2 \int f \left| \frac{\nabla f}{f} + \nabla W \right|^2 + 2 \int f \, |\nabla W - \nabla \tilde{W}|^2$$

and since $|\nabla W(v) - \nabla \widetilde{W}(v)| = \lambda(R)(|v| - R/2)\, 1_{|v| \geqslant R/2} \leqslant \lambda(R)\, |v| 1_{|v| \geqslant R/2}$, we get

$$\int f\, |\nabla W - \nabla \widetilde{W}|^2 \leqslant \lambda(R)^2 \int_{|v| \geqslant R/2} f\, |v|^2\, dv$$
$$\leqslant 2^{s-2} \frac{\lambda(R)^2}{R^{s-2}} M_s(f)$$

Thus,

$$H(f \mid e^{-W}) \leqslant H(f \mid e^{-\widetilde{W}}) + d \int_{|v| > R/2} e^{-W(v)}\, dv$$
$$\leqslant \frac{e^{a+b}}{\lambda(R)} I(f \mid e^{-\widetilde{W}}) + d \int_{|v| > R/2} e^{-W(v)}\, dv$$
$$\leqslant 2 \frac{e^{a+b}}{\lambda(R)} I(f \mid e^{-W}) + C_s(R) \frac{\lambda(R)}{R^{s-2}} M_s(f)$$

where

$$C_s(R) = 2^{s-1} e^{a+b} + d \frac{R^{s-2}}{\lambda(R)} \int_{|v| > R/2} e^{-W(v)}\, dv$$

is a bounded function of R, for all $s \geqslant 2$, in view of the rapid decay of e^{-W}. In fact

$$C_s(R) \leqslant 2^{s-1} e^{a+b} + \frac{d}{c} M_{s-\alpha}(w)$$

Optimizing in R, we find the desired result. ▮

The second part of our program consists in establishing a loose bound on the moments $M_s(f)$, if f is a solution to the Fokker–Planck equation.

Proposition 2. Let f be a solution of (8), with initial datum f_0. Assume that W satisfies for some $\alpha > 0$, $c > 0$, $C_0 > 0$,

$$\nabla W(v) \cdot v \geqslant c\, |v|^{\alpha} - C_0 \tag{17}$$

Then, for all $s \geqslant 2$, and for all time $t \geqslant 0$,

$$M_s(f(t, \cdot)) \leqslant M_s(f_0) + Ct,$$

where C is a constant depending only on α, c, C_0.

Remark. Condition (17) is easily seen to be true if W is C^2 and satisfies $D^2 W \geqslant c' \, |v|^{\alpha-2}$ for $|v|$ (write the Taylor formula of order 1 for W, with endpoints 0 and v).

Proof. Below, C_s will denote various constants depending on s, c, C_0, α.

$$\begin{aligned}
\frac{d}{dt} M_s &= \int f[\Delta(1+|v|^2)^{s+2} - \nabla W(v) \cdot \nabla(1+|v|^2)^{s/2}] \\
&= [Ns + s(s-2)] \, M_{s-2} - s(s-2) \, M_{s-4} \\
&\quad - s \int f[\nabla W(v) \cdot v] \, (1+|v|^2)^{(s-2)/2} \\
&\leqslant C_s M_{s-2} - K M_{s-2+\alpha} + C \\
&\leqslant C_s M_{s-2+\alpha}^{(s-2)/(s-2+\alpha)} - c M_{s-2+\alpha} \leqslant C_s
\end{aligned}$$

Here we have used Hölder's inequality, and the fact that the function $Cx^{\delta} - cx$ is uniformly bounded for $x \geqslant 0$ if $\delta < 1$. This implies the conclusion. ∎

Remark. Let us note immediately that, again by Hölder's inequality, for all $s < u$, one has $M_s \leqslant M_u^{s/u}$, which proves that in fact, if the initial datum has rapid decay (in the sense that $M_s(f_0) < +\infty$ for all $s > 0$), then the growth of the moments is *very slow*:

$$M_s(f(t, \cdot)) \leqslant [M_{s/\varepsilon}(f_0) + C_{s,\varepsilon} t]^{\varepsilon}$$

Yet nothing prevents *all* moments of order $s > 2$ to grow, say, like $\log^{s/2}(1+t)$.

Combining Propositions 1 and 2, we establish the

Theorem 3. Let W be a potential satisfying assumptions (11), (12), and (17). Let f_0 be a probability density such that $H(f_0 \mid e^{-W}) < \infty$, $M_s(f_0) < \infty$ for some $s > 2$, and let $f(t, \cdot)$ be a (smooth) solution of the Fokker–Planck equation (8) with potential W and with initial datum $f(0, \cdot) = f_0$. Then, there is a constant C depending on $H(f_0 \mid e^{-W})$, $M_s(f_0)$ and s, such that for all $t > 0$,

$$\frac{1}{2} \, \|f(t) - e^{-W}\|_{L^1}^2 \leqslant H(f(t) \mid e^{-W}) \leqslant \frac{C}{t^{\kappa}}$$

with

$$\kappa = \frac{1-2\delta}{\delta} = \frac{s-2}{2-\alpha}$$

Proof. We use the shorthands $H = H(f(t) \mid e^{-W})$, $I = I(f(t) \mid e^{-W})$, and we assume that these quantities are positive (if not, there is nothing to prove). It is well known that (under sufficient smoothness), the time-derivative of H is given by I. Applying Propositions 1 and 2, we obtain the differential inequality

$$-\dot{H} = I \geqslant K H^{1/(1-\delta)}(1+t)^{-\delta/(1-\delta)}$$

where K is a constant, or, what is the same,

$$-\dot{H} H^{-1/(1-\delta)} \geqslant K(1+t)^{-\delta/(1-\delta)}$$

Integrating in time from 0 to t, and then inverting the relation, follows

$$H(t) \leqslant \frac{1}{\left\{ H(0)^{-\delta/(1-\delta)} + K \frac{\delta}{1-2\delta} \left[(1+t)^{(1-2\delta)/(1-\delta)} - 1 \right] \right\}^{(1-\delta)/\delta}} \qquad \blacksquare$$

Remarks. 1. In the limit case were $s = 2$, we recover

$$H(t) \leqslant \frac{1}{[H(0)^{-1} + K \log(1+t)]}$$

2. One can also use the fact that if the moment of order M_s is finite at time 0, then the growth of moments of order s', $s' < s$, will be sublinear in time. However, apparently nothing is gained in doing so. In the non-linear case, on the contrary, it will be crucial to "gain" on the rate of growth.

3. THE LANDAU EQUATION FOR MOLLIFIED SOFT POTENTIALS

In this section and in the next, we consider more complicated non-linear models, and we shall not obtain such sharp estimates as in the preceding section.

Here we are interested in the behavior of solutions to

$$\frac{\partial f}{\partial t} = Q_L(f, f) \tag{18}$$

where $Q_L(f, f)$ is given by (3), and

$$c(1+|z|)^{-\beta} \leqslant \frac{\Psi(|z|)}{|z|^2} \leqslant C(1+|z|)^{-\beta}, \qquad 0 < \beta \leqslant 3 \tag{19}$$

More general assumptions are possible, but the important point here is that we deal with a cross-section which is not singular at the origin ($z=0$), since our only aim here is to overcome troubles arising from the degeneracy at infinity. We point out that taking into account a singularity at the origin would (of course!) not worsen the entropy dissipation estimate from below, and would not be a problem either as regards the time behavior of moments. But the time-evolution of Sobolev norms would require further investigation.

Our strategy is exactly the same as before: first, we establish a modified logarithmic Sobolev-type inequality, and then slowly growing a priori bounds on the relevant quantities.

Proposition 4. Let

$$D_L(f) = \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} f f_* \Psi(|v - v_*|) \left| \Pi(v - v_*) \left[\frac{\nabla f}{f} - \left(\frac{\nabla f}{f} \right)_* \right] \right|^2 dv \, dv_*$$

be the entropy dissipation functional. We assume that

$$\Psi(|z|) \geqslant c \, |z|^2 \, (1 + |z|)^{-\beta}, \qquad \beta > 0$$

Then, for all $s > 0$ and all f satisfying the moment condition (6), there is a constant $C_s(f)$, depending on f only through $H(f)$, such that

$$D_L(f) \geqslant C_s(f) \, H(f \mid M)^{1 + (\beta/s)} \, F_s^{-\beta/s}$$

$$F_s = M_{s+2}(f) + J_{s+2}(f),$$

where M_{s+2} is defined by formula (13), and

$$J_{s+2}(f) = \int_{\mathbb{R}^N} |\nabla \sqrt{f}|^2 \, (1 + |v|^2)^{(s+2)/2} \, dv$$

Proof. For all $R>0$, we set

$$\tilde{\Psi}(|z|)=\Psi(|z|)+c(1+R)^{-\beta}\,|z|^2\,1_{|z|>R}$$

Then we apply the main result of ref. 11 to $\tilde{\Psi}$, and find

$$\tilde{D}(f)\equiv\frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} ff_*\,\tilde{\Psi}(|v-v_*|)\left|\Pi(v-v_*)\left[\frac{\nabla f}{f}-\left(\frac{\nabla f}{f}\right)_*\right]\right|^2 dv\,dv_*$$
$$\geqslant K(f)(1+R)^{-\beta}\,I(f\mid M) \qquad (20)$$

where $K(f)$ depends only on $H(f)$.

We then estimate $\tilde{D}$ in terms of D:

$$\tilde{D}(f)\leqslant D(f)+\frac{c}{2(1+R)^\beta}\int_{|v-v_*|\geqslant R} ff_*\,|v-v_*|^2\left|\frac{\nabla f}{f}-\left(\frac{\nabla f}{f}\right)_*\right|^2 dv\,dv_*$$
$$\leqslant D(f)+\frac{4c}{(1+R)^\beta}\left(\int_{|v-v_*|\geqslant R} ff_*\,|v|^2\left|\frac{\nabla f}{f}\right|^2 dv\,dv_*\right.$$
$$\left.+\int_{|v-v_*|\geqslant R} ff_*\,|v_*|^2\left|\frac{\nabla f}{f}\right|^2 dv\,dv_*\right)$$
$$\leqslant D(f)+\frac{16c}{(1+R)^\beta}\,E_s(R)$$

where

$$E_s(R)=\left(\int_{|v|\geqslant R/2} f\,dv\right)\left(\int |\nabla\sqrt{f}|^2\,|v|^2\,dv\right)+\int_{|v|\geqslant R/2} f\,|v|^2\,|\nabla\sqrt{f}|^2\,dv$$
$$+\left(\int_{|v|\geqslant R/2} f\,|v|^2\,dv\right)\left(\int |\nabla\sqrt{f}|^2\,dv\right)$$
$$+\left(\int f\,|v|^2\right)\left(\int_{|v|\geqslant R/2} |\nabla\sqrt{f}|^2\right)$$

Then, with C_s denoting a constant depending on c, s, β, N,

$$\tilde{D}(f)\leqslant D(f)+\frac{C_s}{R^{\beta+s}}\,[M_sJ_2+J_{s+2}+M_{s+2}J_0+J_s]$$

Next, by Hölder and Young inequalities,

$$\begin{aligned} M_s J_2 &= \int f_* \, |\nabla \sqrt{f}|^2 \, (1+|v_*|^2)^{s/2} \, (1+|v|^2) \, dv \, dv_* \\ &\leqslant \left(\int f_* \, |\nabla \sqrt{f}|^2 \, (1+|v_*|^2)^{(s+2)/2} \, dv \, dv_* \right)^{s/(s+2)} \\ &\quad \times \left(\int f_* \, |\nabla \sqrt{f}|^2 \, (1+|v|^2)^{(s+2)/2} \, dv \, dv_* \right)^{2/(s+2)} \\ &= (M_{s+2} J_0)^{s/(s+2)} \, (J_{s+2} M_0)^{2/(s+2)} \leqslant C_s (M_{s+2} J_0 + J_{s+2} M_0) \end{aligned}$$

We also dominate trivially J_s by J_{s+2}, and remain with

$$\tilde{D}(f) \leqslant D(f) + \frac{C_s}{R^{\beta+s}} (M_{s+2} J_0 + J_{s+2})$$

On the whole,

$$D(f) \geqslant \frac{K(f)}{R^\beta} \left[I(f \mid M) - \frac{C_s}{R^s} (M_{s+2} J_0 + J_{s+2}) \right]$$

We now use a simple trick: let $I(f) = \int |\nabla f|^2 / f$ denote the usual Fisher information, and write

$$J_0 = \int_{\mathbb{R}^N} |\nabla \sqrt{f}|^2 \, dv = \tfrac{1}{4} I(f) = \tfrac{1}{4} I(f \mid M) + \tfrac{1}{4} I(M)$$

Thus, if $C_s M_{s+2}/R^s \leqslant 1/2$, one has (for a different constant C_s, in which $I(M) = N$ is taken into account)

$$D(f) \geqslant \frac{K(f)}{R^\beta} \left[\frac{I(f \mid M)}{2} - \frac{C_s}{R^s} (M_{s+2} + J_{s+2}) \right]$$

We conclude by choosing R in such a way $C_s (M_{s+2} + J_{s+2}) \, R^{-s} = I(f \mid M)/4$, and applying the usual logarithmic Sobolev inequality. ∎

Corollary 4.1. Let f be a solution of Eq. (18) with a cross-section $\Psi(|z|) \geqslant c \, |z|^{2-\beta}$, $\beta > 0$, satisfying the moment condition (6). If, for some $s > 0$, one has

$$\begin{cases} M_{s+2}(f(t, \cdot)) + J_{s+2}(f(t, \cdot)) \leqslant C(1+t)^\lambda \\ \dfrac{\lambda \beta}{s} \leqslant 1 \end{cases} \tag{21}$$

then

$$H(f(t,\cdot)\mid M) \xrightarrow[t\to+\infty]{} 0$$

More precisely, $H(f(t,\cdot)\mid M)$ decays at least like $O(t^{-\kappa})$, $\kappa = s/\beta - \lambda$ if $\lambda < s/\beta$, and $O((\ln t)^{-\lambda})$ if $\lambda = s/\beta$.

Proof. This is a consequence of Proposition 4 and the fact that $-\dot{H}(f\mid M) = D_L(f)$. ∎

The next step, obtaining estimates on $F_{s+2}(f)$, is much more delicate. We use the notations

$$\|f\|^2_{H^k_s} = \sum_{|\alpha|\leqslant k} \int (\partial^\alpha f)^2\,(1+|v|^2)^s$$

$$\|f\|^2_{\dot{H}^k_s} = \sum_{|\alpha|=k} \int (\partial^\alpha f)^2\,(1+|v|^2)^s$$

Proposition 5. Let the cross-section $\Psi(|z|)$ satisfy assumption (19) with $0<\beta<2$, and let f be a smooth solution of (18), with initial datum f_0.

- — If $M_{u+2}(f_0) < +\infty$, then for all $s\leqslant u$,

$$M_{s+2}(f)\leqslant C(1+t)^{s/u}$$

- — If $M_{u+2}(f_0)<+\infty$ and $\|f_0\|_{H^2_w}<+\infty$ for some $w=s+2+(N/2)+\varepsilon$, $\varepsilon>0$, with $w-(\beta/2)\leqslant u+2$, then

$$J_{s+2}(f)\leqslant C(1+t)^{\mu}$$

with

$$\mu = \frac{s}{u} + \frac{N-\beta+2\varepsilon}{2u} + \frac{1}{2}$$

Let the cross-section $\Psi(|z|)$ satisfy assumption (19) with $2<\beta<3$.

- — If $M_{u+2}(f_0) < +\infty$, then for all $s\leqslant u$,

$$M_{s+2}(f)\leqslant C(1+t)^{s/3}$$

• — if $M_{u+2}(f_0) < +\infty$ and $\|f_0\|_{H^2_w} < +\infty$ for some $w = s + 2 + (N/2) + \varepsilon$, $\varepsilon > 0$, with $w - (\beta/2) \leqslant u + 2$, then

$$J_{s+2}(f) \leqslant C(1+t)^{\mu}$$

with

$$\mu = \frac{s}{3} + \frac{N - \beta + 2\varepsilon}{6} + \frac{1}{2}$$

Remark. A precise study of regularization properties should enable one to dispend with the conditions $\|f_0\|_{H^2_\alpha} < +\infty$. In particular, it is clear from our estimates that if all L^2_s norms of f are finite at time 0, then all H^1_s, H^2_s norms of f are also finite at all positive times.

Corollary 5.1. Let $\beta \in (0, 3)$, let Ψ be a (smooth) cross-section satisfying (19), and let f_0 be an initial datum satisfying the moment condition (6), and which is rapidly decreasing, in the sense that for all $s > 0$, $\|f_0\|_{L^2_s}$ are finite (this also implies that all the moments of f_0 are finite). Then, for all $\varepsilon > 0$ there is a constant $C_\varepsilon(f_0)$, depending only on ε, N and a finite number of norms $\|f_0\|_{L^2_s}$, such that the (unique) smooth solution f to the Landau equation with cross-section Ψ satisfies

$$H(f(t, \cdot) \mid M) \leqslant C_\varepsilon(f_0)\, t^{-1/\varepsilon}$$

Proof of Corollary 5.1. It suffices to note that as $u \to +\infty$ in the assumptions of Proposition 5, the quantity $s/\beta - \lambda$ goes to $+\infty$ as $s \to +\infty$, in all cases except $\beta = 3$. ∎

Remark. In the case $\beta = N = 3$, our estimate enables a control of M_{s+2} with an exponent that would entail *logarithmic* decay to equilibrium. But the control of J_{s+2} is not sharp enough.

Proof of Proposition 5. We begin with the estimate for M_{s+2}. Let us first recall from ref. 12 the basic equation for the moments $M_s(t) = M_s(f(t, \cdot))$:

$$\frac{d}{dt} M_s(t) = \int_{\mathbb{R}^{2N}} dv\, dv_*\, f f_*\, \frac{\Psi(|v - v_*|)}{|v - v_*|^2}\, (1 + |v|^2)^{(s-2)/2}$$
$$\times [-2(1 + |v|^2) + 2(1 + |v_*|^2)] + (s - 2) \int_{\mathbb{R}^{2N}} dv\, dv_*\, f f_*$$
$$\times \frac{\Psi(|v - v_*|)}{|v - v_*|^2}\, (1 + |v|^2)^{(s-4)/2}\, [\,|v|^2\, |v_*|^2 - (v \cdot v_*)^2] \qquad (22)$$

If $\Psi(|z|)$ satisfies Assumption (19), from this equation follows easily, by standard Hölder-type arguments as in ref. 12,

$$\frac{d}{dt} M_s \leqslant -KM_{s-\beta} + CM_{s-2} \tag{23}$$

for some positive constants C, K.

If $\beta < 2$, then the right-hand side of (23) is bounded (again by Hölder's inequality), and we are left with

$$M_s(t) \leqslant M_s(0) + C(1+t)$$

Using the fact that M_2 is bounded, we also have $M_{s+2} \leqslant CM_{u+2}^{s/u}$, and combined with $M_{u+2} \leqslant C(1+t)$ this proves our claim in Proposition 5.

As long as M_{s+2} is concerned, the criterion (21) is fulfilled with $\lambda = 1$, and any $s > \beta$.

On the other hand, if $\beta \geqslant 2$, from (23) follows only

$$\frac{d}{dt} M_s \leqslant CM_{s-2}$$

and, starting from the energy conservation, this entails

$$M_{s+2}(t) \leqslant C(1+t)^{s/2} \qquad \text{if} \quad M_{s+2}(f_0) < +\infty$$

which is not sufficient (except maybe for $\beta = 2$): note that $(s/2) \times (\beta/2) = \beta/2$, which has to be compared with 1.

We need a more precise bound: to this purpose, we start again from (22). By Hölder's inequality,

$$\begin{aligned}
&\int ff_* \frac{\Psi(|v-v_*|)}{|v-v_*|^2} (1+|v|^2)^{s/2} (1+|v_*|^2) \\
&\quad \leqslant \left[\int ff_* \frac{\Psi(|v-v_*|)}{|v-v_*|^2} (1+|v|^2)^{(s+2)/2} \right]^{s/(s+2)} \\
&\qquad \times \left[\int ff_* \frac{\Psi(|v-v_*|)}{|v-v_*|^2} (1+|v_*|^2)^{(s+2)/2} \right]^{2/(s+2)} \\
&\quad = \int ff_* \frac{\Psi(|v-v_*|)}{|v-v_*|^2} (1+|v|^2)^{(s+2)/2} \, dv \, dv_*
\end{aligned}$$

so that the sum of the first two integrals in the right-hand side of (22) is nonpositive, and we are left with

$$\frac{d}{dt} M_s \leqslant C \int ff_*(1+|v-v_*|)^{-\beta} (1+|v|^2)^{(s-4)/2} [|v|^2 |v_*|^2 - (v \cdot v_*)^2]$$

By the elementary inequality

$$|v|^2 |v_*|^2 - (v \cdot v_*)^2 \leqslant |v| |v_*| |v-v_*|^2$$

and using the fact that $(1+|v-v_*|)^{-\beta} |v-v_*|^2 \leqslant 1$, we find

$$\frac{d}{dt} M_s \leqslant C M_{s-3}$$

Starting from $(d/dt) M_5 \leqslant C$, $(d/dt) M_8 \leqslant C(1+t)$, and so on, then using Hölder's inequality, we find the announced bound for $M_{s+2}(t)$ in $(1+t)^{s/3}$.

We now turn to estimating J_{s+2}. This is a long computation that we divide into several steps, and that we shall not give in full detail.

Step 1: Reduction to Fisher information.
We use the notation $I(g) = 4 \int |\nabla \sqrt{g}|^2$. From the general identity

$$\int |\nabla \sqrt{f}|^2 \varphi = \int |\nabla \sqrt{f\varphi}|^2 + \int f \sqrt{\varphi} \, \Delta \sqrt{\varphi}$$

(easy to obtain by expanding $\int |\nabla(\sqrt{f} \sqrt{\varphi})|^2$ and doing some integrations by parts), we deduce that for any $s > 0$,

$$J_{s+2}(f) = \int |\nabla \sqrt{f}|^2 (1+|v|^2)^{(s+2)/2}$$

$$\leqslant \tfrac{1}{4} I(f(1+|v|^2)^{(s+2)/2}) + C M_s(f)$$

Step 2: Reduction to weighted Sobolev norms.

The diffusive structure of the Landau equation makes it much easier to estimate the evolution of weighted Sobolev norms, than the evolution of Fisher-like functionals. To reduce to Sobolev norms, we shall prove a general functional inequality, which has interest on its own.

Lemma 1. For all $\varepsilon > 0$ there is a constant C_ε, depending only on N and ε, such that the following functional inequality holds,

$$I(g) \leqslant C_\varepsilon (\|g\|_{\dot{H}^2_{(N/2)+\varepsilon}} + \|g\|_{\dot{H}^1_{(N/2)-1+\varepsilon}} + \|g\|_{L^2_{(N/2)-2+\varepsilon}}) \leqslant C_\varepsilon \|g\|_{H^2_{(N/2)+\varepsilon}}$$

Proof of Lemma 1. First, by Cauchy–Schwarz,

$$\int_{\mathbb{R}^N} |\nabla \sqrt{g}|^2 \leqslant \left(\int_{\mathbb{R}^N} |\nabla \sqrt{g}|^4 \, (1+|v|^2)^{(N+\varepsilon)/2} \, dv\right)^{1/2} \left(\int_{\mathbb{R}^N} \frac{dv}{(1+|v|^2)^{(N+\varepsilon)/2}}\right)^{1/2}$$
$$= C_\varepsilon \left(\int_{\mathbb{R}^N} |\nabla \sqrt{g}|^4 \, (1+|v|^2)^{(N+\varepsilon)/2} \, dv\right)^{1/2}$$

Then, by the inequality

$$|\nabla a|^4 \, b^4 \leqslant 8 \, |\nabla (ab)|^4 + 8a^4 \, |\nabla b|^4$$

follows

$$\int_{\mathbb{R}^N} |\nabla \sqrt{g}|^4 \, (1+|v|^2)^{(N+\varepsilon)/2} \, dv$$
$$\leqslant 8 \left(\int |\nabla \sqrt{g(1+|v|^2)^{(N+\varepsilon)/4}}|^4 \, dv + \int |\sqrt{g} \, \nabla \sqrt{(1+|v|^2)^{(N+\varepsilon)/4}}|^4 \, dv\right)$$
$$\leqslant C\left(\int |\nabla \sqrt{g(1+|v|^2)^{(N+\varepsilon)/4}}|^4 \, dv + \int g^2 (1+|v|^2)^{(N-4+\varepsilon)/2}\right)$$

At this point, we apply the inequality

$$\|\nabla \sqrt{a}\|_{L^4}^4 \leqslant C \, \|a\|_{\dot{H}^2}^2$$

which is extracted from ref. 14, and recover

$$\int |\nabla \sqrt{g(1+|v|^2)^{(N+\varepsilon)/4}}|^4$$
$$\leqslant C \int |D^2 [\, g(1+|v|^2)^{(N+\varepsilon)/4}]|^2$$
$$\leqslant C\left(\int |D^2 g|^2 \, (1+|v|^2)^{(N+\varepsilon)/2} + \int |\nabla g|^2 \, (1+|v|^2)^{(N-2+\varepsilon)/2}\right.$$
$$\left. + \int g^2 (1+|v|^2)^{(N-4+\varepsilon)/2}\right)$$
$$\leqslant C(\|g\|^2_{\dot{H}^2_{(N+\varepsilon)/2}} + \|g\|^2_{\dot{H}^1_{(N-2+\varepsilon)/2}} + \|g\|^2_{L^2_{(N-4+\varepsilon)/2}})$$

from which the conclusion follows. ∎

Combining Steps 1 and 2, we obtain

$$J_{s+2}(f) \leqslant C(\|f\|_{H^2_{s+2+(N/2)+\varepsilon}} + M_s(f))$$

Step 3: We show that if f is a smooth solution of (18), then

$$\frac{d}{dt}\|f\|^2_{H^2_s} \leqslant CM_{s-(\beta/2)}(f)^2$$

We first handle $\|f\|^2_{L^2_s}$.

We use the notations $b = \nabla \cdot a$, $c = \nabla \cdot b$, and $\bar{a} = a * f$ ($\bar{a}$ is a matrix!), $\bar{b} = b * f$, $\bar{c} = c * f$, so that the Landau equation can be rewritten as

$$\partial_t f = \nabla \cdot (\bar{a}\,\nabla f - \bar{b} f)$$

We recall the following bounds from ref. 12, valid as soon as f has finite mass, energy and entropy:

$$K(1+|v|)^{-\beta}\, I_N \leqslant \bar{a} \leqslant C(1+|v|)^{2-\beta}\, I_N \tag{24}$$

(here I_N stands for the identity matrix of order N, and this is an inequality in the sense of matrices),

$$|\bar{b}| \leqslant C(1+|v|)^{1-\beta}, \qquad |\bar{c}| \leqslant C(1+|v|)^{-\beta} \tag{25}$$

Here and below, C denotes various finite constants, and K various positive constants.

We then write

$$\begin{aligned}
\frac{d}{dt}\int f^2\varphi &= 2\int f\,\nabla\cdot(\bar{a}\,\nabla f - \bar{b}f)\,\varphi \\
&= 2\int \bar{a}\,\nabla f\,\nabla f\,\varphi + 2\int \bar{b}f\,\nabla f\,\varphi - 2\int f(\bar{a}\,\nabla f - \bar{b}f)\,\nabla\varphi \\
&= -2\int \bar{a}\,\nabla f\,\nabla f\,\varphi + \int f^2[-\bar{c}\varphi - 4\bar{b}\cdot\nabla\varphi - \bar{a} : D^2\varphi]
\end{aligned}$$

where we have used $2f\,\nabla f = \nabla(f^2)$, $\nabla\cdot\bar{a} = \bar{b}$, $\nabla\cdot b = \bar{c}$, and performed as many integrations by parts as necessary.

Choosing then $\varphi = (1+|v|)^s$ and using the bounds (24) and (25), we obtain

$$\frac{d}{dt}\int f^2(1+|v|^2)^s \leqslant -K\int |\nabla f|^2\,(1+|v|^2)^{s-(\beta/2)} + C\int f^2(1+|v|^2)^{s-(\beta/2)}$$

Writing once again $|\nabla f|^2\,\varphi^2 \leqslant 2\,|\nabla(f\varphi)|^2 + 2f^2\,|\nabla\varphi|^2$, this is bounded by

$$-K\int|\nabla g|^2 + C\int g^2$$

where $g = f(1+|v|^2)^{(s/2)-(\beta/4)}$. We now use another interpolation lemma.

Lemma 2. For all $\varepsilon > 0$, there is a constant C_ε such that the following functional inequality holds,

$$\|g\|_{L^2}^2 \leqslant \varepsilon\,\|g\|_{\dot H^1}^2 + C_\varepsilon\,\|g\|_{L^1}^2$$

Proof of Lemma 2. Let $\hat g(\xi)$ denote the Fourier transform of $g(v)$. For all $R > 0$ we have

$$\begin{aligned}\int_{\mathbb{R}^N}|g(v)|^2\,dv &= \int_{\mathbb{R}^N}|\hat g(\xi)|^2\,d\xi\\ &= \int_{|\xi|\leqslant R}|\hat g(\xi)|^2\,d\xi + \frac{1}{R^2}\int_{|\xi|>R}|\xi|^2\,|\hat g(\xi)|^2\,d\xi\\ &\leqslant |B_N(R)|\;\|g\|_{L^1}^2 + \frac{1}{R^2}\,\|g\|_{\dot H^1}^2\end{aligned}$$

where $B_N(R)$ is the ball of radius R in $\mathbb{R}^N$. Choosing $R = 1/\sqrt{\varepsilon}$ we finish the proof. ∎

From Lemma 2 follows

$$\frac{d}{dt}\,\|f\|_{L_s^2}^2 \leqslant CM_{s-(\beta/2)}(f)^2$$

Next, we differentiate equation (18) with respect to v_k, to find

$$\partial_t\partial_k f = \nabla\cdot(\bar a\,\nabla\partial_k f - \bar b\,\partial_k f) + \nabla\cdot(\partial_k\bar a\,\nabla f - \partial_k\bar b f)$$

Writing the equation for $(d/dt)\int(\partial_k f)^2\,\varphi$ and playing with integration by parts and Cauchy–Schwarz inequalities, we find

$$\begin{aligned}\frac{d}{dt}\,\|f\|_{\dot H_s^1}^2 &\leqslant -K\,\|g\|_{\dot H^2}^2 + C\,\|g\|_{\dot H^1}^2 + C\,\|g\|_{L^2}^2\\ &\leqslant -K\,\|g\|_{\dot H^2}^2 + C\,\|g\|_{L^2}^2\end{aligned}$$

with $g = f(1+|v|^2)^{(s-\beta)/4}$. By a lemma similar to Lemma 2, we deduce

$$\frac{d}{dt}\|f\|^2_{\dot{H}^1_s} \leqslant CM_{s-(\beta/2)}(f)^2$$

The same proof, only still more tedious, shows that

$$\frac{d}{dt}\|f\|^2_{\dot{H}^1_s} \leqslant CM_{s-(\beta/2)}(f)^2$$

Step 4: We can now conclude the proof of Proposition 5. We assume that $\|f_0\|_{H^2_w} < +\infty$ for some $w = s+2+N/2+\varepsilon$, $\varepsilon > 0$, such that $w-\beta/2 \leqslant u+2$. In the case $\beta < 2$, we write

$$\frac{d}{dt}\|f\|^2_{H^2_w} \leqslant CM_{w-(\beta/2)}(f) \leqslant C(1+t)^{2((2w-\beta-4)/2u)}$$

$$\|f\|_{H^2_w} \leqslant C(1+t)^{((2w-\beta-4)/2u)+(1/2)}$$

and

$$J_{s+2}(f) \leqslant C(1+t)^{\mu}$$

$$\mu = \max\left(\frac{s-2}{u}, \frac{s}{u} + \frac{N-\beta+2\varepsilon}{2u} + \frac{1}{2}\right) = \frac{s}{u} + \frac{N-\beta+2\varepsilon}{2u} + \frac{1}{2}$$

A similar computation is done for $\beta \leqslant 3$, and yields

$$\mu = \max\left(\frac{s-2}{3}, \frac{s}{3} + \frac{N-\beta+2\varepsilon}{6} + \frac{1}{2}\right) = \frac{s}{3} + \frac{N-\beta+2\varepsilon}{6} + \frac{1}{2} \quad \blacksquare$$

Remark. If one would like to extend this result to $\beta = N = 3$, there are several possible strategies. One is to try to be more keen at the level of the interpolations analogous to Lemma 2, as regards the terms involving weighted H^1 and H^2 norms ("diagonal" interpolation). Another one is to work directly with the functionals

$$I(g) = \int \frac{|\nabla g|^2}{g}, \qquad K(g) = \sum_{ij} \int \left(\frac{\partial_{ij} g}{g} - \frac{\partial_i g\, \partial_j g}{g^2}\right)^2 g$$

which appear naturally in a Fokker–Planck context, see ref. 18. It is likely that an interpolation inequality like $I(g) \leqslant \varepsilon K(g) + C_\varepsilon H(g)$ can be proved, maybe by some semigroup regularization argument.

4. THE BOLTZMANN EQUATION FOR MOLLIFIED SOFT POTENTIALS

In this section, we consider the case of the Boltzmann equation

$$\frac{\partial f}{\partial t} = Q(f, f) \tag{26}$$

where Q is given by (4). Without loss of generality, we deal with probability densities satisfying the moment condition (6), and the entropy functional $H(f \mid M) = H(f) - H(M)$.

Our aim is to illustrate our general strategy, and not to get sharp results. Therefore, we shall make several simplifying assumptions, in particular that the cross-section is reasonably smooth and locally bounded below. Taking into account possible vanishing of the kernel is easy but tedious.[7, 11] From now on, we assume that

$$B(z, \sigma) = \Phi(|z|)\, b(k \cdot \sigma), \qquad k = \frac{z}{|z|}, \qquad 0 < \underline{b} \leqslant b(k \cdot \sigma) \leqslant \bar{b} < +\infty \tag{27}$$

$$K_0(1 + |z|)^{-\beta} \leqslant \Phi(|z|) \leqslant C_0(1 + |z|)^{-\beta} \tag{28}$$

and we denote by

$$D(f) = \frac{1}{4} \int_{\mathbb{R}^N \times \mathbb{R}^N \times S^{N-1}} dv\, dv_*\, d\sigma\, B(v - v_*, \sigma)(f'f'_* - ff_*) \log \frac{f'f'_*}{ff_*}$$

the entropy dissipation associated to the Boltzmann equation.

We follow the same strategy as before. From ref. 19 we extract the following *rough* bound, obtained by keeping track of the constants. Below, we use the notation

$$\|f\|_{L^1_s \log L} = \int_{\mathbb{R}^N} f \log(1 + f)(1 + |v|^2)^{s/2}$$

Proposition 6. Assume that f satisfies the moment condition (6) and $f \geqslant Ke^{-A|v|^2}$, and let $D(f)$ be the entropy dissipation associated with the kernel B, where $B(z, \sigma) \geqslant K(1 + |v|)^{-\beta}$. Then

$$D(f) \geqslant C(f)\, H(f \mid M)^{1 + ((2+\beta)/s)}\, F_s^{-(2+\beta)/s}$$

where $C(f)$ depends only on the entropy $H(f)$, and

$$F_s = \left(\log \frac{1}{K} + A \right) \|f\|^2_{L^1_{5+s}} \|f\|_{L^1_{3+s} \log L}$$

An easy modification of the proof yields a comparable result (only involving higher moments of f) if we only assume that $f \geqslant K e^{-A |v|^p}$ for some $p \geqslant 2$.

Clearly, to control $\|f\|_{L^1_u \log L}$ it is sufficient to control both moments and L^2 norm of f. It is easy to obtain bounds on moments, by a method similar to the one sketched in the preceding section, as long as we stay in the range of *moderately soft potentials*. In fact, one can prove the

Proposition 7. Let f be a solution of the Boltzmann equation (26), with a cross-section satisfying (28), $\beta < 2$. Then, all the moments of f increase at most linearly in time, i.e.

$$M_s(f) \leqslant C(1+t) \qquad \text{if} \quad M_s(f_0) < +\infty.$$

We admit this proposition here, and we refer to ref. 10 for a complete proof in the case $\beta \leqslant 1$.

Corollary 7.1. Let f be a solution of the Boltzmann equation with a cross-section satisfying (28), $\beta < 2$, and an initial datum f_0 with finite moments of all orders ($\forall u > 0$, $M_u(f_0) < +\infty$). Then, for all s, ε, there is a constant C, depending only on s, ε and a finite number of moments of f_0, such that

$$M_s(f) \leqslant C(1+t)^{\varepsilon}$$

This corollary is an immediate consequence of Proposition 7 and Hölder's inequality.

Next, we turn to the control of some L^p norm. Again, we are not interested in sharp results, and we only wish to show that explicit L^p estimates can be obtained with a simple method, based on the so-called Q^+ smoothness properties. We extract the following estimate from ref. 22 (or rather the proof of the main result there).

Proposition 8. Let

$$Q^+(g, f) = \int_{\mathbb{R}^N} B(v - v_*, \sigma)\, g'_*\, f'$$

where B satisfies assumptions (27) and (28) (in particular, Φ and b are bounded). Assume in addition that B is smooth (C^1, uniformly). Then, there is a universal constant (depending on N, B), such that for all $f \in L^2_1(\mathbb{R}^N)$,

$$\|Q^+(g, f)\|_{\dot{H}^{(N-1)/2}} \leqslant C\, \|g\|_{L^1_1} (\|f\|_{L^1_1} + \|f\|_{L^2_1}) \tag{29}$$

From now on, we drop the (constant) term $\|f\|_{L^1_1}$ and absorb it in the constant C, since $\|f\|_{L^2_1}$ is bounded below. Thus, we shall write

$$\|Q^+(g, f)\|_{\dot{H}^{(N-1)/2}} \leqslant C^+\, \|f\|_{L^2_1} \tag{30}$$

On the other hand, let

$$Q^-(f, f) = \int B(v - v_*, \sigma)\, ff_*\, dv_*\, d\sigma = Cf(f * \Phi)$$

It is classical that if Φ satisfies (28), then

$$Q^-(f, f) \geqslant K^- f(1 + |v|^2)^{-\beta/2} \tag{31}$$

where K^- is a constant depending only on the mass, energy and entropy of f (see ref. 2 for instance).

With the help of these two auxiliary estimates, we shall prove the

Proposition 9. Let f be a (strong) solution of the Boltzmann equation with a cross-section B satisfying (30) and (31). Let $p \in (1, 2N)$. Then, there exist finite positive constants C, s, α, depending only on N, p and the constants C^+, K^- in (30), (31), such that

$$\frac{d}{dt} \|f\|_{L^p_p} \leqslant CM_s(f)^\alpha$$

Corollary 9.1. If in addition the initial datum f_0 has all its moments finite, and satisfies $\|f_0\|_{L^p} < +\infty$ for some $p \in (1, 2N)$, then for all $\varepsilon > 0$ there is a constant C_ε such that

$$\|f\|_{L^p_p} \leqslant C_\varepsilon (1 + t)^{1/\varepsilon}$$

Proof of Proposition 9. We use the notation

$$\|f\|^p_{L^p_s} = \int_{\mathbb{R}^N} f^p (1 + |v|^2)^{ps/2}, \qquad 1 \leqslant p < \infty, \quad s \in \mathbb{R}$$

We denote by p' the conjugate exponent to p, i.e., $p' = p/(p-1)$. Let $q \in (1, p')$ be an auxiliary exponent. We compute

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int f^p &= \int f^{p-1}Q^+(f,f) - \int f^{p-1}Q^-(f,f)\\ &\leqslant \|f^{p-1}\|_{L^q}\,\|Q^+(f,f)\|_{L^{q'}} - K - \int \frac{f^p}{(1+|v|^2)^{\beta/2}}\\ &= \|f\|^{p-1}_{L^{q(p-1)}}\,\|Q^+(f,f)\|_{L^{q'}} - K_-\,\|f\|^p_{-\beta/p}\end{aligned} \tag{32}$$

Now, we estimate $\|Q^+(f,f)\|_{L^{q'}}$. From (30) and Sobolev injection there is a constant C such that

$$\|Q^+(g,f)\|_{L^{2N}} \leqslant C\,\|g\|_{L^1_1}\,\|f\|_{L^2_1}$$

On the other hand, using the boundedness of B, it is easy to prove (by duality)

$$\|Q^+(g,f)\|_{L^1} \leqslant C\,\|g\|_{L^1}\,\|f\|_{L^1}$$

Since Q^+ is a bilinear operator, we can interpolate between these two bounds, and we get

$$\|Q^+(f,g)\|_{L^{q'}} \leqslant C\,\|g\|_{L^1_1}\,\|f\|_{L^1_1} \tag{33}$$

with

$$\frac{1}{r} = 1 - \frac{1/(2q)}{1-1/(2N)}$$

Next, we recall that by the Stein–Weiss interpolation theorem (see also ref. 15),

$$\|f\|_{L^p_k} \leqslant \|f\|^{\theta}_{L^{p_1}_{k_1}}\,\|f\|^{1-\theta}_{L^{p_2}_{k_2}}$$

if

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \qquad k = \theta k_1 + (1-\theta)\,k_2$$

In particular,

$$\|f\|_{L^r_1} \leqslant \|f\|^{\theta}_{L^p_{-\beta/p}}\,\|f\|^{1-\theta}_{L^1_s} \tag{34}$$

with

$$\theta = \left(\frac{p'}{q}\right) \frac{1/2}{1-1/(2N)} \in (0, 1), \qquad s = \frac{1+(\theta\beta/p)}{1-\theta}$$

Also,

$$\|f\|_{L^{q(p-1)}} \leqslant \|f\|^{\mu}_{L^p_{-\beta/p}} \|f\|^{1-\mu}_{L^1_w} \tag{35}$$

with

$$\mu = \frac{1-1/q(p-1)}{1-1/p} \in (0, 1), \qquad w = \frac{\mu\beta}{(1-\mu)\,p}$$

(here we used the fact that $q < p'$).

Combining (32), (33), (34), (35), we find after some computation

$$\frac{d}{dt} \int f^p \leqslant C\, \|f\|^{\alpha}_{L^1_\kappa} (\|f\|^p_{L^p_{-\beta/p}})^{1-\delta} - K^- \, \|f\|^p_{L^p_{-\beta/p}}$$

with $\kappa < +\infty$ and

$$\delta = \frac{1}{q(p-1)} \left(\frac{N-1}{2N-1}\right) > 0$$

The conclusion then follows from the elementary inequality

$$\forall X \geqslant 0, \qquad AX^{1-\delta} - KX \leqslant CA^{1/\delta} \quad \blacksquare$$

In order to control F_s in Proposition 6, it only remains to estimate A and $\log K^{-1}$ such that

$$f \geqslant K e^{-A\,|v|^2}$$

Again, we consider a very simple situation. It is clear that our proof works just the same with the assumption $f \geqslant K e^{-A\,|v|^p}$.

Proposition 10. Assume that f is a solution of the Boltzmann equation, with a cross-section B uniformly bounded, and initial datum.

$$f_0 \geqslant K_0 e^{-A_0\,|v|^2}$$

Then,

$$f \geqslant K e^{-A\,|v|^2}$$

with

$$A(t) = A_0, \qquad \log \frac{1}{K(t)} \leqslant \log \frac{1}{K_0} + Ct, \qquad C = |S^{N-1}|\ \|B\|_{L^\infty}\ \|f_0\|_{L^1}$$

Proof. Let $M(z) = \int_{S^{N-1}} B(z, \sigma)\, d\sigma$, and $Lf = f * M$. Clearly, $\|M\|_{L^\infty} \leqslant |S^{N-1}|\ \|B\|_{L^\infty}$.

We write the solution f in Duhamel representation:

$$\begin{aligned} f(t, v) &= f_0(v)\, e^{-\int_0^t Lf(\tau, v)\, d\tau} + \int_0^t Q^+(f(s, v), f(s, v))\, e^{-\int_s^t Lf(\tau, v)\, d\tau}\, ds \\ &\geqslant K_0 e^{-A_0\,|v|^2} e^{-t\,\|Lf\|_{L^\infty}} \end{aligned}$$

The conclusion follows from

$$\|Lf\|_{L^\infty} \leqslant \|f\|_{L^1}\ \|M\|_{L^\infty} \quad \blacksquare$$

Combining Propositions 7, 9, 10, we obtain that under suitable decay and positivity assumptions on the initial datum, F_s in Proposition 6 satisfies a bound like, say, $C(1+t)^2$. Since the exponent does not depend on s, we can choose s very large, and find a bound on the evolution of the relative entropy, H, like $-\dot{H} \geqslant K(1+t)^{-\eta} H^{1+\varepsilon}$ with η and ε small numbers. It is then easy to conclude with the following theorem (in which, again, the assumption $f_0 \geqslant K_0 e^{-A_0\,|v|^2}$ can be replaced by $f_0 \geqslant K_0 e^{-A_0\,|v|^q}$ for some $q < +\infty$).

Theorem 11. Let f be a solution of the Boltzmann equation with a smooth (C^1, uniformly) cross-section satisfying assumption (28), $\beta < 2$, and an initial datum f_0 such that f_0 has finite moments of all orders, $\|f_0\|_{L^p} < +\infty$ for some $p > 1$, $f_0 \geqslant K_0 e^{-A_0\,|v|^2}$ for some finite positive constants K_0, A_0. Then, for all $\varepsilon > 0$ there is a constant $C_\varepsilon(f_0)$, depending only on N, p, ε, $\|f_0\|_{L^p}$, K_0, A_0, B and a finite number of moments of f_0, such that

$$H(f(t, \cdot) \mid M) \leqslant C_\varepsilon t^{-1/\varepsilon}$$

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